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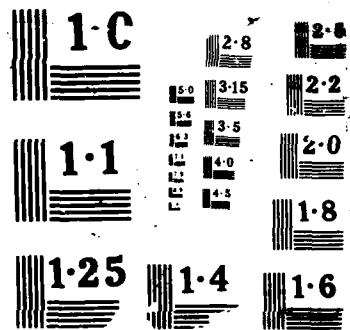
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ON THE BAROTROPIC MODEL OF THE OCEAN CIRCULATION

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Abstract. We examine the mathematical properties of the solutions of a barotropic, wind driven ocean with bottom friction on both a β - and f -plane. Except for small Rossby numbers, the uniqueness of the solutions of the corresponding partial differential equations dependent on an *a priori* bound for the gradient of the vorticity. For the f -plane, two drivings are considered which give rise to explicit, global unique solutions. For large Rossby numbers, a novel nonlocal, nonlinear boundary value problem, which does depend on the β -effect, is obtained for the circulation.

1. Introduction. This paper is concerned with the question of whether ocean circulation models have unique steady solutions. This question is very important for understanding the long-term behaviour of these models. More specifically, we shall consider this question for the simplest such model, namely that of a homogeneous wind-driven ocean, with bottom friction and no topography.

For the readers unfamiliar with the oceanographic jargon, the above ocean circulation model assumes a simplified dynamics embodied in the following set of equations:

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$$\left. \begin{aligned} u_t + uu_x + vu_y - fv &= -\rho p_x - Ku - \tau_1, \\ v_t + uv_x + vv_y + fu &= -\rho p_y - Kv - \tau_2, \\ u_x + v_y &= 0. \end{aligned} \right\} \quad (1.1)$$

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These equations govern the two-dimensional motion of a homogeneous fluid viewed from a rotating frame of reference. The usual oceanographic notations are used, namely u and v represent the velocity components in the x and y directions which point eastward and northward respectively; ρ is the density; p the dynamical pressure, and (τ_1, τ_2) the wind stresses in the eastern and northern direction. The Coriolis parameter f is equal to $f_0 + \beta y$. Finally, K is a Rayleigh friction coefficient. Since there are many instances where a detailed analysis of the effects of the bottom viscous boundary later give rise to Rayleigh-like friction terms, this type of dissipation is referred to as "bottom friction". Thus, these equations include all the physical processes which are believed to be important, namely inertial forces, Coriolis force, dissipation and wind forcing. In spite of the fact that these equations constitute an enormous oversimplification of the oceanographic reality, mathematically they are still very difficult to deal with because of their nonlinearity.

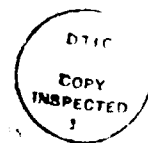
In order to avail ourselves of the *a-priori* estimates for the strength of the circulation, for the magnitude of the currents, and for the vorticity, which we previously derived (Barcilon et al., 1987), we switch to the following dimensionless formulation of the problem:

$$\left. \begin{aligned} R(\psi_x \omega_y - \psi_y \omega_x) + \psi_x + \epsilon \omega &= F \\ \nabla^2 \psi &= \omega \end{aligned} \right\} \quad \text{in } \Omega \quad (1.2a)$$

with

$$\psi = 0 \quad \text{on } \partial\Omega \quad (1.2b)$$

The notations are identical to those of Veronis (1966a,b) whose early numerical study of this problem has been very inspirational to us. The Rossby number R represents a relative measure of the inertial effects; ϵ , which is akin to the Ekman number, represents a relative



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measure of frictional effects. The circulation is described by the streamfunction ψ and the vorticity ω . The only differences with Veronis's work are (i) that the curl of the wind stress, which we denote by F , is taken to be a general function of x and y rather than $\sin x \sin y$ and (ii) the basin, denoted here by Ω with coastline $\partial\Omega$, is taken to have an arbitrary shape rather than the square $0 < x, y < \pi$.

2. Uniqueness: We show in this section how the uniqueness hinges crucially upon whether or not the gradient of the vorticity is bounded, or in mathematical terms, upon whether we can get an *a priori* estimate of $\|\nabla\omega\|_\infty$. As usual, $\|\cdot\|_\infty$ stands for the $L^\infty(\Omega)$ norm of a function, namely

$$\|h\|_\infty = \sup_{\Omega} |h|$$

Let (ψ, ω) and (ψ', ω') be two solutions of (1.2). If we write:

$$\psi' = \psi + \phi \tag{2.1a}$$

and

$$\omega' = \omega + \eta \tag{2.1b}$$

then clearly

$$\left. \begin{aligned} RJ(\psi, \eta) + RJ(\phi, \omega) + RJ(\phi, \eta) + \phi_x + \epsilon\eta &= 0 \\ \nabla^2 \phi &= \eta \end{aligned} \right\} \tag{2.2}$$

where $J(f, g)$ stands for the Jacobian $f_x g_y - f_y g_x$. Multiplying (2.2a) by ϕ and

integrating over the basin, we deduce that:

$$R \int_{\Omega} \phi J(\psi, \eta) dA + \int_{\Omega} \phi \phi_x dA + \epsilon \int_{\Omega} \phi \nabla^2 \phi dA = 0 \quad (2.3)$$

since

$$\begin{aligned} \int_{\Omega} \phi J(\phi, h) dA &= \int_{\Omega} \left\{ \left(\frac{1}{2} \phi^2 h_y \right)_x + \left(-\frac{1}{2} \phi^2 h_x \right)_y \right\} dA \\ &= \int_{\partial\Omega} \{ n_1 h_y - n_2 h_x \} ds \\ &= 0 \end{aligned}$$

in view of the fact that ϕ vanishes on the boundary. In the above formulas, h is an arbitrary function, $\mathbf{n} = (n_1, n_2)$ is the outer unit normal and s is the arc length along the coast. Making use of the divergence theorem on the last two terms, (2.3) can be written as

$$\epsilon \|\nabla \phi\|^2 = - \int_{\Omega} \phi \nabla \cdot (\eta \mathbf{Q}) dA \quad (2.4)$$

where

$$\mathbf{Q} \equiv (u, v) = (-\psi_y, \psi_x) \quad (2.5)$$

is the velocity field associated with ψ . The notation $\| \cdot \|$ is the customary shorthand for the $L_2(\Omega)$ norm of a function, namely

$$\|h\|^2 = \int_{\Omega} |h|^2 dA$$

The term on the left hand side of (2.4) can be transformed as follows. First, we use the divergence theorem once again to get

$$\int_{\Omega} \phi \nabla \cdot (\eta \mathbf{Q}) dA = - \int_{\Omega} \eta \mathbf{Q} \cdot \nabla \phi dA$$

then, taking absolute values of both sides we immediately see that

$$|\int_{\Omega} \phi \nabla \cdot (\eta \mathbf{Q}) dA| \leq \|\nabla \psi\|_{\infty} \int_{\Omega} |\eta| |\nabla \phi| dA$$

In deriving this formula we have used the obvious fact that the modulus of \mathbf{Q} and that of $\nabla \psi$ are identical. The last step consists is using Cauchy's inequality to overestimate the right hand side in the last equation. This implies that

$$|\int_{\Omega} \phi \nabla \cdot (\eta \mathbf{Q}) dA| \leq \|\nabla \psi\|_{\infty} \|\eta\| \|\nabla \phi\|$$

Substituting this estimate in (2.4), we conclude that

$$\epsilon \|\nabla \phi\| \leq R \|\nabla \psi\|_{\infty} \|\eta\| \quad (2.6)$$

Similarly, multiplying the first equation in (2.2) by η and integrating, we see that

$$R \int_{\Omega} \eta J(\phi, \omega) dA + \int_{\Omega} \eta \phi_x dA + \epsilon \|\eta\|^2 = 0$$

because

$$\begin{aligned} \int_{\Omega} \eta J(h, \eta) dA &= \int_{\Omega} \{h_x (\frac{1}{2} \eta^2)_y - h_y (\frac{1}{2} \eta^2)_x\} dA \\ &= \int_{\Omega} (\{h (\frac{1}{2} \eta^2)_y\}_x - \{h (\frac{1}{2} \eta^2)_x\}_y) dA \\ &= \int_{\partial \Omega} \{(\frac{1}{2} \eta^2)_y n_1 - (\frac{1}{2} \eta^2)_x n_2\} ds \\ &= 0 \end{aligned}$$

if h stands for either ϕ or ψ which both vanish on the coast. Applying as before Cauchy's inequality, we see that,

$$\epsilon \|\eta\| \leq (R \|\nabla \omega\|_{\infty} + 1) \|\nabla \phi\| \quad (2.7)$$

Combining the two inequalities, we get:

$$\|\nabla\phi\| \leq \frac{R}{\epsilon^2} \|\nabla\psi\|_{\infty} (1 + R\|\nabla\omega\|_{\infty}) \|\nabla\phi\| \quad (2.8)$$

Thus, if

$$\frac{R}{\epsilon^2} \|\psi\|_{\infty} (1 + R\|\nabla\omega\|_{\infty}) \leq 1 \quad (2.9)$$

$\nabla\phi$ and hence ϕ and η are identically zero, i.e. the solution is unique. We should recall at this stage that $\|\psi\|_{\infty}$ contains ϵ . Unfortunately, we do not have an expression for $\|\nabla\omega\|_{\infty}$. Therefore, (2.9) need not coincide with the regime dominated by dissipation.

The uniqueness in this viscously dominated regime, can be deduced by writing (1.2) as an integral equation, namely;

$$\psi = \int_{\Omega} F G dA + R \int_{\Omega} \nabla^2 \psi J(\psi, G) dA \quad (2.10)$$

where G is the Green function

$$\epsilon \nabla^2 G - G_x = \delta(\mathbf{r} - \mathbf{r}') \quad (2.11)$$

For small R , we can show by classical arguments that a power series in R converges.

In view of the non linearity of the problem, it would not be surprising if the viscous, Stommel like solution were to bifurcate for a certain value of R . In the next section, we shall consider the f -plane problem, which though very different oceanographically from the β -plane problem, is in fact rather similar mathematically. In particular, progress on the question of uniqueness is also hampered by the lack of an *a priori* bound on the gradient of vorticity. Therefore, it is surprising to see meaningful wind stresses yielding unique circulations for all values of R .

3. The f -plane problem: The analog of equations (1.2) are

$$\left. \begin{aligned} R(\psi_x \omega_y - \psi_y \omega_x) + \epsilon \omega &= F \\ \nabla^2 \psi &= \omega \end{aligned} \right\} \quad \text{in } \Omega \quad (3.1a)$$

with

$$\psi = 0 \quad \text{on } \partial\Omega \quad (3.1b)$$

As mentioned previously, the uniqueness derivation leads to an inequality similar to (2.8), namely

$$\epsilon^2 \geq R^2 \|\nabla \psi\|_\infty \|\nabla \omega\|_\infty \quad (3.2)$$

The reason for considering the f -plane problem is that because of its simplicity, we are able to exhibit two solutions valid for all R 's. These solutions are the responses to the forcings $F = -1$, and $F = -u_1$, where u_1 is the first eigenfunction of the Laplacian for the basin. Incidentally, it is ironic that Veronis used the forcing $F = \sin x \sin y$, which is in fact the first eigenfunction for the square basin $0 < x, y < \pi$! However, his analysis was for the β -plane.

We start with the case $F = -1$, i.e. of a constant wind curl. If we set

$$\omega = -\epsilon^{-1} \quad (3.3)$$

then the vorticity equation is identically satisfied. The streamfunction is found by solving

$$\nabla^2 \psi = -\epsilon^{-1} \quad (3.4)$$

subject to the boundary condition along the coast. For this particular solution $\|\nabla \omega\|_\infty$ is identically equal to zero. Hence, this is the unique solution valid for all Rossby numbers.

Incidentally, the lack of dependence on the Rossby number should be noted and it should be recalled that all our estimates on the norms of ψ and ω obtained in Barcilon et al. (1987) were independent of R .

The case $F = -u_1$, where

$$\left. \begin{aligned} \nabla^2 u_n + \lambda_n u_n &= 0, & \text{in } \Omega \\ u_n &= 0 & \text{on } \partial\Omega \end{aligned} \right\} \quad (3.5)$$

is more subtle. Once again we can check that the following expressions satisfy the equations (3.1).

$$\left. \begin{aligned} \psi &= -\frac{u_n(x, y)}{\epsilon \lambda_n}, \\ \omega &= \frac{u_n(x, y)}{\epsilon} \end{aligned} \right\} \quad (3.6)$$

We then go through the uniqueness procedure and write

$$\left. \begin{aligned} \psi &= -\frac{u_n(x, y)}{\epsilon \lambda_n} + \phi, \\ \omega &= \frac{u_n(x, y)}{\epsilon} + \eta \end{aligned} \right\} \quad (3.7)$$

By means of some simple manipulations we see that:

$$\frac{R}{\epsilon \lambda_n} (u_n, J(\phi, \eta)) - \epsilon \|\phi\|^2 = 0,$$

and

$$-\frac{R}{\epsilon} (u_n, J(\phi, \eta)) + \epsilon \|\eta\|^2 = 0$$

This implies that

$$\|\phi\| = \lambda_n \|\eta\| \quad (3.8)$$

i.e. that ϕ is a linear combination of the eigenfunctions associated with λ_n . Thus, the multiplicity of solutions associated with the driving $F = u_n$ is equal to the multiplicity of

the eigenvalue λ_n . Interestingly, the first eigenvalue has multiplicity one (Protter 1987).

Thus, for $F = -u_1$, the solution exhibited above is unique for all R 's.

4. Large Rossby number asymptotics: If we multiply the governing equation (1.2) by $(\psi_x \omega_y - \psi_y \omega_x)$ which we denote for brevity by J , then clearly

$$R \|J\|^2 = -(J, \psi_x) + (J, f)$$

and since (Barcilon et al., 1987)

$$\|\nabla \psi\| \leq 2^{1/2} \|f\| (1 + \frac{1}{\epsilon^2 \lambda_1})^{1/2}$$

it follows that

$$\|J\| \leq \frac{\|f\|}{R} \{1 + 2^{1/2} (1 + \frac{1}{\epsilon^2 \lambda_1})^{1/2}\} \quad (4.1)$$

Thus, as the Rossby number increases there is a tendency for the vorticity to remain constant along streamlines.

In view of the above tendency, we look for a solution as an asymptotic series in inverse powers of R of the form

$$\left. \begin{aligned} \psi &= \Psi + \frac{1}{R} \psi^{(1)} + \dots \\ \omega &= Z(\Psi) + \frac{1}{R} \omega^{(1)} + \dots \end{aligned} \right\} \quad (4.2)$$

Also, to avoid unnecessary complications at this stage, we restrict ourselves to wind drivings which are negative throughout the entire basin. This insures the existence of a single gyre. To leading order, the dynamical equation (1.2) is identically satisfied. To the next order, we get:

$$J(\Psi, \omega^{(1)}) + J(\psi^{(1)}, Z(\Psi)) + \Psi_x + \epsilon Z = F \quad (4.3)$$

If ds represents an element of arc length along a streamline, then after integrating (4.3) around a streamline, we deduce that:

$$Z(\Psi) = \epsilon^{-1} \frac{\oint \frac{F ds}{|\nabla \Psi|}}{\oint \frac{ds}{|\nabla \Psi|}} \quad (4.4)$$

This determines the distribution of vorticity to leading order, provided that Ψ is known. Note that the β -effect is not felt to that order. In order to find Ψ , we must solve the following boundary value problem:

$$\left. \begin{aligned} \nabla^2 \Psi &= \epsilon^{-1} \frac{\oint \frac{F ds}{|\nabla \Psi|}}{\oint \frac{ds}{|\nabla \Psi|}} && \text{in } \Omega, \\ \Psi &= 0, && \text{on } \partial\Omega \end{aligned} \right\} \quad (4.5)$$

This boundary value problem is highly non standard: not only is it nonlinear but also non local. Similar problems have been encountered by Batchelor (1956) and Rhines & Young (1983) among others. In both of these papers, the authors were interested in determining the vorticity distribution in a fluid region with closed streamlines. We are not aware of any mathematical analysis of such problems. In section 5, we reformulate this problem in terms of natural coordinates in the hope that they will help future analyses. For the time being, we assume that Ψ has been found and that it is unique, and proceed with the formal asymptotic expansion.

For convenience, we define the average \hat{h} of a function h as follows:

$$\hat{h} = \frac{\oint \frac{h ds}{|\nabla \Psi|}}{\oint \frac{ds}{|\nabla \Psi|}} \quad (4.6)$$

Using now

$$\mathbf{Q} = (-\Psi_y, \Psi_x) \quad (4.7)$$

to denote the leading order velocity, we rewrite (4.3) thus:

$$-Q \frac{\partial}{\partial s} \{ \omega^{(1)} - (dZ/d\Psi) \psi^{(1)} + y \} = F - \hat{F} \quad (4.8)$$

In the above equation, we have explicitly taken into account the fact that F is negative and hence that the flow around any streamline is in the clockwise direction whereas s is measured in the counterclockwise one. Also, we wrote Q for the magnitude of \mathbf{Q} . Clearly

$$\omega^{(1)} + y - (dZ/d\Psi) \psi^{(1)} = \int_0^s \frac{(F - \hat{F}) ds}{Q} + C(\Psi) \quad (4.9)$$

In order to determine $C(\Psi)$, we turn to the second order equation, viz.

$$J(\Psi, \omega^{(2)}) + J(\psi^{(1)}, \omega^{(1)}) + J(\psi^{(2)}, F(\Psi)) + \psi_x^{(1)} + \epsilon \omega^{(1)} = 0 \quad (4.10)$$

or equivalently

$$-Q \frac{\partial}{\partial s} \{ \omega^{(1)} - (dZ/d\Psi) \psi^{(1)} \} + J(\psi^{(1)}, \omega^{(1)} + y) + \epsilon \omega^{(1)} = 0 \quad (4.11)$$

Dividing by Q and integrating around $\Psi = \text{constant}$ lines, we see that

$$\oint Q^{-1} J(\psi^{(1)}, \omega^{(1)} + y) ds + \oint Q^{-1} \epsilon \omega^{(1)} ds = 0$$

which after we make use of (4.9) implies that

$$\begin{aligned} C(\Psi) = & - \frac{\oint Q^{-1} J(\psi^{(1)}, \int_0^s Q^{-1} (F - \hat{F}) ds)}{\oint Q^{-1} ds} \\ & + \epsilon \frac{\oint Q^{-1} [-y + (dZ/d\Psi) \psi^{(1)} + \int_0^s Q^{-1} (F - \hat{F})] ds}{\oint Q^{-1} ds} \end{aligned} \quad (4.12)$$

We can now obtain an equation for $\psi^{(1)}$ if we recall that

$$\nabla^2 \psi^{(1)} = \omega^{(1)}$$

We close this section by returning to the two special forcings discussed previously, namely the constant wind curl $F = -1$ and the wind curl equal to the first eigenfunction of the Laplacian for the ocean basin $F = -u_1$.

For $F = -1$, we immediately deduce that $\hat{F} = -1$. Therefore, (4.5) simplifies greatly and becomes:

$$\left. \begin{aligned} \nabla^2 \Psi &= -\epsilon^{-1} & \text{in } \Omega, \\ \Psi &= 0, & \text{on } \partial\Omega \end{aligned} \right\} \quad (4.13)$$

For $F = -u_1$, we can see that the assumption that Ψ is proportional to u_1 does not lead to any inconsistencies. Indeed, under this assumption, the problem for Ψ reduces to

$$\left. \begin{aligned} \nabla^2 \Psi &= -\epsilon^{-1} u_1 & \text{in } \Omega, \\ \Psi &= 0, & \text{on } \partial\Omega \end{aligned} \right\} \quad (4.14)$$

and consequently

$$\Psi = -\frac{u_1}{\epsilon \lambda_1} \quad (4.15)$$

5. The problem for Ψ : We return to the problem for the leading term in the asymptotic series, namely

$$\left. \begin{aligned} \nabla^2 \Psi &= \hat{F} & \text{in } \Omega \\ \Psi &= 0 & \text{on } \partial\Omega \end{aligned} \right\} \quad (5.1)$$

We restrict our attention to forcings of one sign, say $F < 0$. To simplify the right hand side, we switch to the natural coordinates Ψ and s , where s is the arc length. The unknowns are $X(s, \Psi)$ and $Y(s, \Psi)$ which provide a parametric representation of the streamlines. Define the Jacobian

$$I = \frac{\partial(X, Y)}{\partial(s, \Psi)} \quad (5.2)$$

For future reference, we record the following formulas:

$$\left. \begin{aligned} \Psi_x &= -I^{-1}Y_s \\ \Psi_y &= I^{-1}X_s \end{aligned} \right\} \quad \left\{ \begin{aligned} s_x &= I^{-1}Y_\Psi \\ s_y &= -I^{-1}X_\Psi \end{aligned} \right.$$

As a result, we can deduce the expression for the Laplacian of Ψ

$$\nabla^2 \Psi = I^{-1} \left(-Y_s \frac{\partial}{\partial \Psi} + Y_\Psi \frac{\partial}{\partial s} \right) (-I^{-1}Y_s) + I^{-1} \left(X_s \frac{\partial}{\partial \Psi} - X_\Psi \frac{\partial}{\partial s} \right) (-I^{-1}X_s),$$

or better still

$$\nabla^2 \Psi = \frac{I_\Psi}{I^3} - (X_s X_\Psi + Y_s Y_\Psi) \frac{I_s}{I^3} - (X_\Psi X_{ss} + Y_\Psi Y_{ss}) \frac{1}{I^2} \quad (5.3)$$

In the process of deriving the above expression, we have made use of the fact that

$$(4.4) \quad X_s^2 + Y_s^2 = 1$$

From the definition (4.1) of I , we have

$$I = X_s Y_\Psi - X_\Psi Y_s \quad (5.5)$$

Finally, we note that

$$Q = \sqrt{\Psi_x^2 + \Psi_y^2} = I^{-1}$$

As a result, the problem (5.1) is transformed into the solution of the three coupled partial differential equations

$$\left. \begin{aligned} \frac{I_\Psi}{I^3} - (X_s X_\Psi + Y_s Y_\Psi) \frac{I_s}{I^3} - (X_\Psi X_{ss} + Y_\Psi Y_{ss}) \frac{1}{I^2} &= \epsilon^{-1} \frac{\int I F ds}{\int I ds}, \\ I &= X_s Y_\Psi - X_\Psi Y_s, \\ X_s^2 + Y_s^2 &= 1. \end{aligned} \right\} \quad (5.6)$$

6. Discussion: We have seen that the lack of *a priori* bound on the L^∞ norm of the gradient of vorticity prevented us from obtaining a uniqueness result valid beyond the viscous, small R regime. At this stage, we do not know whether such a bound is just difficult to derive or simple impossible, i.e. whether one can envisage wind curl which lead to very large values of the gradient of vorticity.

The same remarks apply to the f -plane analog. However, in this case, we saw that for flows such as those driven by a wind curl which is either constant or proportional to the first eigenfunction of the basin, the vorticity gradient is bounded and the solution is unique for all values of the Rossby number. Incidentally, for these two drivings the maximum value of the vorticity was in the ocean rather than along the coast as was the case in the numerical calculations of Veronis (1966b) for the β -plane.

For large values of R , we have derived a non-local problem for the leading order streamfunction. The vorticity is then roughly constant on these streamlines. The associated circulation shows a very weak East-West asymmetry.

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